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The Hilbert Expansion to the Boltzmann Equation for Steady Flow

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The Hilbert expansion to the Boltzmann equation is carried out for steady flow. It is shown that the first term in the Hilbert series for the distribution function is a local Maxwellian leading to the steady Euler equations. The steady field equations that follow from the solution of the second term in the series are derived. The formulas for thermal conductivity and for viscosity of Hilbert that appear in the steady field equations of the second approximation are shown to be precisely the same as those obtained by Chapman and Enskog. The procedure to obtain higher approximations by Hilbert's method is summarized.

KEY WORDS: The Hilbert expansion; the Boltzmann equation; the local Maxwellian; the solvability conditions; the summational invariants.

1. INTRODUCTION

The properties of a binary gas are usually characterized by the well-known Boltzmann equation. Although the equation was proposed by Boltzmann(1) more than a century ago, the analytical methods of solution are few owing to the nonlinear complex structure of the collision integral.

A class of approximate analytical solutions called normal solutions can be constructed assuming that the distribution function f does not explicitly depend on the coordinates $\mathbf{x}$ and time t. The distribution function f can then be expanded in the form

$$f = f_L(1 + \Phi^{(1)} + \Phi^{(2)} + \cdots) \tag{1.1}$$

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where f_L is a local Maxwellian given by

$$f_L = n\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left[-\frac{m}{2T}(\mathbf{v}-\mathbf{u})^2\right] \tag{1.2}$$

and n is the number density, $\mathbf{u}$ is the gross velocity field, T is the temperature field (measured in energy units), $\mathbf{v}$ is the atomic (or molecular) velocity, and m is the atomic (or molecular) mass of the species constituting the gas. f_L itself does not satisfy the Boltzmann equation unless n, $\mathbf{u}$, and T are constants (the uniform state of the gas) and the functions $\Phi^{(1)}$, $\Phi^{(2)}$, etc. are determined so as to make the distribution function f a solution of the Boltzmann equation. The hydrodynamic equations for n, $\mathbf{u}$, and T can then be obtained. The procedure summarized above is due to Chapman(3) and Enskog(5) and the expansion (1.1) is called the Chapman–Enskog expansion.

Historically, the Chapman–Enskog theory was proceeded by a slightly different expansion due to Hilbert.(8,9) As will be discussed in detail in Section 2 of this paper, the Hilbert expansion is constructed by introducing an auxiliary parameter which may be set equal to unity at the end of the calculations and by expanding both the distribution function f and the hydrodynamic variables n, $\mathbf{u}$, and T in powers of this parameter.

Hilbert himself succeeded in showing that the first term in the series for the distribution function f was the local Maxwellian. Thus, the derived hydrodynamic equation were Euler's equations of ideal gas flow as will be discussed in Section 3 of this work. An attempt to obtain the second term in the Hilbert series for the distribution function f leading to a computation of the coefficients of thermal conductivity and of viscosity was made by Lunn.(11),[3] In the published abstract of Lunn's paper, there is strong evidence to believe that Lunn succeeded in obtaining the general form of the second term that appears in the Hilbert series and ended in formulas for thermal conductivity and viscosity. Unfortunately, Lunn's paper never appeared in print; thus, the details of his computations are lost. Since then, to the best of this author's knowledge, no work has been published in this direction, probably because of the Chapman–Enskog procedure which appeared later leading to evaluations of thermal conductivity and viscosity.

The aim of this paper is to carry on the Hilbert expansion to the Boltzmann equation from the state where it was left. Although the details of the Hilbert method and questions about the convergence of the Hilbert series can be found in the works of Grad,(6,7) Kogan,(10) Cercignani,(2) and McLennan(12), discussions of the second and higher terms in the series are seldom given.

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[3] The author learned about Lunn's work from the referees of this paper.

In this study, the Hilbert series is carried on beyond the first term. In doing so, the expansion is restricted to steady flow to avoid the initial value problem. The steady field equations (the Navier–Stokes equations) that follow from the second approximation are derived. The coefficients of thermal conductivity and viscosity of Hilbert are shown to be precisely the same as those obtained by Chapman and Enskog. The procedure to carry out the expansion to obtain higher approximations is also discussed.

2. THE HILBERT METHOD OF EXPANSION

Let κ be an auxiliary expansion parameter. In the absence of an external field, the Boltzmann equation can be written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \frac{1}{\kappa} J[f] \tag{2.1}$$

where $J[f]$ is the collision integral given by

$$J[f] = \int [f'f_1' - ff_1]\, g\, d\sigma\, d^3v_1 \tag{2.2}$$

and

$$f' = f(\mathbf{v}'), \qquad f_1' = f(\mathbf{v}_1'), \qquad f = f(\mathbf{v}), \qquad f_1 = f(\mathbf{v}_1)$$

and

$$g = |\mathbf{v} - \mathbf{v}_1|$$

$\mathbf{v}$, $\mathbf{v}_1$ and $\mathbf{v}'$, $\mathbf{v}_1'$ are the velocities before and after a binary encounter, respectively, and $d\sigma$ is the differential cross-section.

To construct a normal solution to the Boltzmann equation by the Hilbert expansion, the distribution function f is expanded in powers of κ about f_L, which will be identified as the local Maxwellian:

$$f = f_L(1 + \kappa\phi^{(1)} + \kappa^2\phi^{(2)} + \kappa^3\phi^{(3)} + \cdots) \tag{2.3}$$

Similarly, the number density n, the gross velocity $\mathbf{u}$, and the temperature T are expanded in powers of κ:

$$n = n^{(0)} + \kappa\delta n + \kappa^2\delta n' + \kappa^3\delta n'' + \cdots \tag{2.4}$$

$$\mathbf{u} = \mathbf{u}^{(0)} + \kappa\delta\mathbf{u} + \kappa^2\delta\mathbf{u}' + \kappa^3\delta\mathbf{u}'' + \cdots \tag{2.5}$$

and

$$T = T^{(0)} + \kappa\delta T + \kappa^2\delta T' + \kappa^3\delta T'' + \cdots \tag{2.6}$$

Each term in the above expansions for the hydrodynamic variables n, $\mathbf{u}$, and T can be related to the terms in the Hilbert series for the distribution function f [Eq. (2.3)] by substitution from Eqs. (2.3), (2.4), (2.5), and (2.6) into the defining relations

$$n = \int f d^3v \tag{2.7}$$

$$nu_i = \int v_i f d^3v \tag{2.8}$$

$$\tfrac{3}{2}nT + \tfrac{1}{2}mnu^2 = \int \tfrac{1}{2}mv^2 f d^3v \tag{2.9}$$

and by collecting the terms of equal power in κ together. The parameter κ which appears in the above expansions is introduced for collecting terms together belonging to the same approximation and may be set equal to unity at the end of the calculations.

Substituting from Eq. (2.3) into Eq. (2.2), one has

$$\begin{aligned} J[f] = J[f_L] - n^{(0)} f_L \Big\{ & \kappa I\phi^{(1)} + \kappa^2\big[I\phi^{(2)} + J_s(\phi^{(1)},\phi^{(1)})\big] \\ & + \kappa^3\big[I\phi^{(3)} + J_s(\phi^{(1)},\phi^{(2)}) + J_s(\phi^{(2)},\phi^{(1)})\big] \\ & + \cdots \Big\} \end{aligned} \tag{2.10}$$

where

$$n^{(0)} J_s(\phi,\psi) = \int f_{L1}(\phi\psi_1 - \phi'\psi_1') g \, d\sigma \, d^3v_1 \tag{2.11}$$

and

$$n^{(0)} I\phi = \int f_{L1}(\phi + \phi_1 - \phi' - \phi_1') g \, d\sigma \, d^3v_1 \tag{2.12}$$

If Eqs. (2.3) and (2.10) are substituted into Eq. (2.1), equal powers of κ will yield

$$J[f_L] = 0 \tag{2.13}$$

$$\frac{\partial f_L}{\partial t} + \mathbf{v} \cdot \nabla f_L = -n^{(0)} f_L I\phi^{(1)} \tag{2.14}$$

$$\frac{\partial (f_L\phi^{(1)})}{\partial t} + \mathbf{v} \cdot \nabla (f_L\phi^{(1)}) = -n^{(0)} f_L\big[I\phi^{(2)} + J_s(\phi^{(1)},\phi^{(1)})\big] \tag{2.15}$$

$$\frac{\partial (f_L\phi^{(2)})}{\partial t} + \mathbf{v} \cdot \nabla (f_L\phi^{(2)}) = -n^{(0)} f_L\big[I\phi^{(3)} + J_s(\phi^{(1)},\phi^{(2)}) + J_s(\phi^{(2)},\phi^{(1)})\big] \tag{2.16}$$

etc. It follows from Eq. (2.13) that f_L is a local Maxwellian. Equations (2.14), (2.15), and (2.16) then constitute a set of uncoupled inhomogeneous linear integral equations for $\phi^{(1)}$, $\phi^{(2)}$, and $\phi^{(3)}$. From the theory of inhomogeneous linear integral equations, nontrivial solutions for $\phi^{(1)}$, $\phi^{(2)}$,

$\phi^{(3)}$, etc. exist only if the solvability conditions

$$\int \psi_\alpha\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f_L \, d^3v = 0 \tag{2.17}$$

$$\int \psi_\alpha\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f_L \phi^{(1)} \, d^3v = 0 \tag{2.18}$$

$$\int \psi_\alpha\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f_L \phi^{(2)} \, d^3v = 0 \tag{2.19}$$

$$\int \psi_\alpha\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f_L \phi^{(3)} \, d^3v = 0 \tag{2.20}$$

etc. are satisfied where ψ_α is any of the summational invariants 1, $\mathbf{v} - \mathbf{u}^{(0)}$, and $\epsilon \equiv \frac{1}{2} m (\mathbf{v} - \mathbf{u}^{(0)})^2$.

To avoid problems of the initial state of the gas, from now on the discussion is restricted to the case of steady flow. In this case, all of the partial time derivatives disappear.

3. THE FIRST APPROXIMATION (THE EULER EQUATIONS)

The first term in the Hilbert series for the distribution function satisfies Eq. (2.13). Thus, f_L is a local Maxwellian. By utilizing Eqs. (2.7), (2.8), and (2.9) which yield

$$n^{(0)} = \int f_L \, d^3v \tag{3.1}$$

$$n^{(0)} u_i^{(0)} = \int v_i f_L \, d^3v \tag{3.2}$$

$$\tfrac{3}{2} n^{(0)} T^{(0)} + \tfrac{1}{2} m n^{(0)} u^{(0)^2} = \int \tfrac{1}{2} m v^2 f_L \, d^3v \tag{3.3}$$

in the first approximation, the local Maxwellian f_L can be written as

$$f_L = n^{(0)} \left(\frac{m}{2\pi T^{(0)}}\right)^{3/2} \exp\left[-\frac{m}{2T^{(0)}} (\mathbf{v} - \mathbf{u}^{(0)})^2\right] \tag{3.4}$$

The solvability conditions (2.17) for steady flow then yield the steady flow Euler equations:

$$\frac{\partial}{\partial x_i} (\rho^{(0)} u_i^{(0)}) = 0 \tag{3.5}$$

$$\rho^{(0)} u_j^{(0)} \frac{\partial u_i^{(0)}}{\partial x_j} = -\frac{\partial P^{(0)}}{\partial x_i} \tag{3.6}$$

$$u_j^{(0)} \frac{\partial T^{(0)}}{\partial x_j} = -\frac{2}{3} T^{(0)} \frac{\partial u_k^{(0)}}{\partial x_k} \tag{3.7}$$

where the pressure and mass density in the first approximation are given by

$$p^{(0)} = n^{(0)}T^{(0)} \tag{3.8}$$

and

$$\rho^{(0)} = mn^{(0)} \tag{3.9}$$

respectively.

4. THE SECOND APPROXIMATION (THE NAVIER-STOKES EQUATIONS)

The next approximation in the Hilbert series for the distribution function satisfies Eq. (2.14). By substitution from Eq. (3.4) into (2.14) and utilizing Eqs. (3.5), (3.6), and (3.7), one obtains

$$\begin{aligned} -n^{(0)}I\phi^{(1)} &= \frac{1}{T^{(0)2}}\left[\frac{1}{2}m(\mathbf{v}-\mathbf{u}^{(0)})^2 - \frac{5}{2}T^{(0)}\right]\left(v_i - u_i^{(0)}\right)\frac{\partial T^{(0)}}{\partial x_i} \\ &\quad + \frac{m}{T^{(0)}}\left[\left(v_i - u_i^{(0)}\right)\left(v_j - u_j^{(0)}\right) - \frac{1}{3}\delta_{ij}(\mathbf{v}-\mathbf{u}^{(0)})^2\right]\frac{\partial u_i^{(0)}}{\partial x_j} \end{aligned} \tag{4.1}$$

The operator I in Eq. (4.1) is a linear integral operator. To simplify the manipulations, let $\mathscr{A}_i$ and τ_{ij} be functions defined by

$$I\mathscr{A}_i = \left[\tfrac{1}{2}m(\mathbf{v}-\mathbf{u}^{(0)})^2 - \frac{5}{2}T^{(0)}\right]\left(v_i - u_i^{(0)}\right) \tag{4.2}$$

$$I\tau_{ij} = m\left[\left(v_i - u_i^{(0)}\right)\left(v_j - u_j^{(0)}\right) - \tfrac{1}{3}\delta_{ij}(\mathbf{v}-\mathbf{u}^{(0)})^2\right] \tag{4.3}$$

satisfying the orthogonality conditions

$$\int \psi_\alpha \mathscr{A}_i f_L \, d^3v = 0 \tag{4.4}$$

$$\int \psi_\alpha \tau_{ij} f_L \, d^3v = 0 \tag{4.5}$$

The solution of the linear integral equations (4.2) and (4.3) for $\mathscr{A}_i$ and τ_{ij} satisfying the orthogonality conditions (4.4) and (4.5) can be determined by expressing them in terms of certain polynomials called Sonine polynomials (for details, one may consult Chapman and Cowling$^{(4)}$). Approximate solutions of $\mathscr{A}_i$ and τ_{ij} can be obtained by a variational technique (such a technique is discussed in McLennan$^{(12)}$).

Once $\mathscr{A}_i$ and τ_{ij} are determined, the solution for $\phi^{(1)}$ can be immediately written down as

$$\phi^{(1)} = -\frac{1}{n^{(0)}T^{(0)2}}\mathscr{A}_i\frac{\partial T^{(0)}}{\partial x_i} - \frac{1}{n^{(0)}T^{(0)}}\tau_{ij}D_{ij} + \psi^{(1)} \tag{4.6}$$

where

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i^{(0)}}{\partial x_j} + \frac{\partial u_j^{(0)}}{\partial x_i}\right) \tag{4.7}$$

and $\psi^{(1)}$ is a linear form of the summational invariants given by

$$\psi^{(1)} = C_1 + C_\epsilon \epsilon + C_i\left(v_i - u_i^{(0)}\right) \tag{4.8}$$

The coefficients C_1, C_ϵ, and $\mathbf{C} = (C_x, C_y, C_z)$ are functions of x, y, and z only and are related to the hydrodynamic variables through Eqs. (2.7), (2.8), and (2.9), which after some manipulations reduce to

$$\delta n = \int f_L \psi^{(1)}\, d^3v = n^{(0)}\left(C_1 + \tfrac{3}{2} T^{(0)} C_\epsilon\right) \tag{4.9}$$

$$n^{(0)} \delta u_i = \int \left(v_i - u_i^{(0)}\right) f_L \psi^{(1)}\, d^3v = n^{(0)} \frac{T^{(0)}}{m} C_i \tag{4.10}$$

$$\tfrac{3}{2}\left(T^{(0)} \delta n + n^{(0)} \delta T\right) = \int f_L \epsilon \psi^{(1)}\, d^3v = n^{(0)}\left(\tfrac{3}{2} T^{(0)} C_1 + \tfrac{15}{4} T^{(0)^2} C_\epsilon\right) \tag{4.11}$$

in the second approximation. In obtaining relations (4.9), (4.10), and (4.11), the orthogonality conditions (4.4) and (4.5) for ν_i and τ_{ij} together with the integrals (A1), (A2), and (A6) given in the Appendix are utilized. Solving from Eqs. (4.9), (4.10), and (4.11) for C_1, C_i, and C_ϵ, one obtains

$$C_1 = \frac{\delta n}{n^{(0)}} - \frac{3}{2} \frac{\delta T}{T^{(0)}} \tag{4.12}$$

$$C_i = \frac{m}{T^{(0)}} \delta u_i \tag{4.13}$$

$$C_\epsilon = \frac{\delta T}{T^{(0)^2}} \tag{4.14}$$

To obtain the hydrodynamic equations in this approximation, one only needs to write down the solvability conditions for steady flow

$$\int \psi_\alpha \mathbf{v} \cdot \nabla\left(f_L \phi^{(1)}\right) d^3v = 0 \tag{4.15}$$

and carry out the manipulations for $\psi_\alpha = 1$, $v_i - u_i^{(0)}$, and $\frac{1}{2} m (v_i - u_i^{(0)})^2$.

(i) $\psi_\alpha = 1$. In this case, the solvability conditions (4.15) become

$$\frac{\partial}{\partial x_i} \int v_i f_L \phi^{(1)}\, d^3v = 0 \tag{4.16}$$

Substitution from Eq. (4.6) into Eq. (4.16) for $\phi^{(1)}$ and utilizing the orthogonality conditions (4.4) and (4.5) for ν_i and τ_{ij} yield

$$\frac{\partial}{\partial x_i}\left[\int \left(v_i - u_i^{(0)}\right) f_L \psi^{(1)}\, d^3v + u_i^{(0)} \int f_L \psi^{(1)}\, d^3v\right] = 0 \tag{4.17}$$

Utilizing the integrals (A1) and (A2) given in the Appendix, one obtains

$$\frac{\partial}{\partial x_i}\left[n^{(0)}\frac{T^{(0)}}{m}C_i + \left(n^{(0)}C_1 + \frac{3}{2}n^{(0)}T^{(0)}C_\epsilon\right)u_i^{(0)}\right] = 0 \qquad (4.18)$$

(ii) $\psi_\alpha = v_i - u_i^{(0)}$. If ψ_α is taken to be $v_i - u_i^{(0)}$, the solvability conditions (4.15) can be written in the form

$$\frac{\partial}{\partial x_j}\left[\int(v_i - u_i^{(0)})(v_j - u_j^{(0)})f_L\phi^{(1)}\,d^3v + u_j^{(0)}\int(v_i - u_i^{(0)})f_L\phi^{(1)}\,d^3v\right]$$
$$+ \frac{\partial u_i^{(0)}}{\partial x_j}\left[\int(v_j - u_j^{(0)})f_L\phi^{(1)}\,d^3v + u_j^{(0)}\int f_L\phi^{(1)}\,d^3v\right] = 0 \qquad (4.19)$$

Substitution from Eq. (4.6) into Eq. (4.19) and utilizing the orthogonality conditions (4.4) and (4.5) for ς_i and τ_{ij} once more yield

$$\frac{\partial}{\partial x_j}\left\{\int(v_i - u_i^{(0)})(v_j - u_j^{(0)})f_L\left[-\frac{1}{n^{(0)}T^{(0)^2}}\varsigma_k\frac{\partial T^{(0)}}{\partial x_k} - \frac{1}{n^{(0)}T^{(0)}}\tau_{rs}D_{rs}\right.\right.$$
$$\left.\left.+\psi^{(1)}\right]d^3v + u_j^{(0)}\int(v_i - u_i^{(0)})f_L\psi^{(1)}\,d^3v\right\}$$
$$+ \frac{\partial u_i^{(0)}}{\partial x_j}\left[\int(v_j - u_j^{(0)})f_L\psi^{(1)}\,d^3v + u_j^{(0)}\int f_L\psi^{(1)}\,d^3v\right] = 0 \qquad (4.20)$$

If the integrals (A1), (A2), (A3), (A4), and (A5) given in the Appendix are substituted into Eq. (4.20), one obtains

$$\frac{\partial}{\partial x_j}\left[-\frac{\eta}{m}\left(\frac{\partial u_i^{(0)}}{\partial x_j} + \frac{\partial u_j^{(0)}}{\partial x_i} - \frac{2}{3}\frac{\partial u_k^{(0)}}{\partial x_k}\delta_{ij}\right)\right.$$
$$\left.+\left(n^{(0)}\frac{T^{(0)}}{m}C_1 + \frac{5}{2}n^{(0)}\frac{T^{(0)^2}}{m}C_\epsilon\right)\delta_{ij} + u_j^{(0)}n^{(0)}\frac{T^{(0)}}{m}C_i\right]$$
$$+ \frac{\partial u_i^{(0)}}{\partial x_j}\left[n^{(0)}\frac{T^{(0)}}{m}C_j + \left(n^{(0)}C_1 + \frac{3}{2}n^{(0)}T^{(0)}C_\epsilon\right)u_j^{(0)}\right] = 0 \qquad (4.21)$$

where η is the dynamic viscosity defined by Eq. (A5). Equation (A5) can be put in a more convenient form by utilizing the orthogonality conditions (4.5) for τ_{ij} and setting $i = k$ and $j = l$ to yield the Hilbert formula for the dynamic viscosity

$$\eta = \frac{1}{10n^{(0)}T^{(0)}}\int f_L\tau_{ij}(I\tau_{ij})\,d^3v \qquad (4.22)$$

where the linear integral operator I is given by Eq. (2.12) and τ_{ij} is to be solved from the linear integral equation (4.3). The Hilbert formula for the dynamic viscosity given by Eq. (4.22) is precisely the same as that obtained by Chapman and Enskog.

(iii) $\psi_\alpha = \epsilon \equiv \frac{1}{2} m(\mathbf{v} - \mathbf{u}^{(0)})^2$. In this case, the solvability conditions (4.15) after some manipulations reduce to

$$\frac{\partial}{\partial x_i}\left[\int \epsilon (v_i - u_i^{(0)}) f_L \phi^{(1)}\, d^3v + u_i^{(0)} \int \epsilon f_L \psi^{(1)}\, d^3v\right]$$
$$+ m \frac{\partial u_j^{(0)}}{\partial x_i}\left[\int (v_j - u_j^{(0)})(v_i - u_i^{(0)}) f_L \phi^{(1)}\, d^3v\right.$$
$$\left. + u_i^{(0)} \int (v_j - u_j^{(0)}) f_L \psi^{(1)}\, d^3v\right] = 0 \qquad (4.23)$$

By substituting from Eq. (4.6) into Eq. (4.23) and utilizing the integrals given in the Appendix, it is not hard to show

$$\frac{\partial}{\partial x_i}\left[-\lambda \frac{\partial T^{(0)}}{\partial x_i} + \frac{5}{2} n^{(0)} \frac{T^{(0)^2}}{m} C_i + u_i^{(0)}\left(\frac{3}{2} n^{(0)} T^{(0)} C_1 + \frac{15}{4} n^{(0)} T^{(0)^2} C_\epsilon\right)\right]$$
$$+ \frac{\partial u_k^{(0)}}{\partial x_k}\left(\frac{2}{3}\eta \frac{\partial u_l^{(0)}}{\partial x_l} + n^{(0)} T^{(0)} C_1 + \frac{5}{2} n^{(0)} T^{(0)^2} C_\epsilon\right)$$
$$- \eta\left(\frac{\partial u_i^{(0)}}{\partial x_j} \frac{\partial u_j^{(0)}}{\partial x_i} + \frac{\partial u_i^{(0)}}{\partial x_j} \frac{\partial u_i^{(0)}}{\partial x_j}\right) + n^{(0)} T^{(0)} C_j u_i^{(0)} \frac{\partial u_j^{(0)}}{\partial x_i} = 0 \qquad (4.24)$$

where the dynamic viscosity η is given by Eq. (4.22) and λ is the thermal conductivity defined by Eq. (A8). The thermal conductivity λ can be explicitly related to the law of intermolecular force and other effects by utilizing the orthogonality conditions (4.4) for ς_i and setting $i = j$ in Eq. (A8). One then obtains

$$\lambda = \frac{1}{3 n^{(0)} T^{(0)^2}} \int f_L \varsigma_i (I \varsigma_i)\, d^3v \qquad (4.25)$$

where I is the linear integral operator defined by Eq. (2.12) and ς_i is to be solved from the linear integral equation (4.2). The Hilbert formula for thermal conductivity given by Eq. (4.25) is again precisely the same as that obtained by Chapman and Enskog.

To write down the steady field equations in the second approximation by the Hilbert method, one only needs to substitute for the coefficients C_1, C_ϵ, and C_i from Eqs. (4.12), (4.13), and (4.14) into Eqs. (4.18), (4.21), and

(4.24). The steady field equations which one arrives at are

$$\frac{\partial}{\partial x_j}\left(\rho^{(0)}\delta u_j + u_j^{(0)}\delta\rho\right) = 0 \tag{4.26}$$

$$\frac{\partial}{\partial x_j}\left[(\delta P)\delta_{ij} + u_i^{(0)}u_j^{(0)}\delta\rho + \rho^{(0)}u_i^{(0)}\delta u_j + \rho^{(0)}u_j^{(0)}\delta u_i - \eta\left(\frac{\partial u_i^{(0)}}{\partial x_j} + \frac{\partial u_j^{(0)}}{\partial x_i} - \frac{2}{3}\frac{\partial u_k^{(0)}}{\partial x_k}\delta_{ij}\right)\right] = 0 \tag{4.27}$$

$$\frac{\partial}{\partial x_i}\left(-\lambda\frac{\partial T^{(0)}}{\partial x_i}\right) + \frac{2}{3}\eta\left(\frac{\partial u_k^{(0)}}{\partial x_k}\right)^2 - \eta\left(\frac{\partial u_i^{(0)}}{\partial x_j}\frac{\partial u_j^{(0)}}{\partial x_i} + \frac{\partial u_i^{(0)}}{\partial x_j}\frac{\partial u_i^{(0)}}{\partial x_j}\right) + \frac{3}{2}n^{(0)}\left(u_i^{(0)}\frac{\partial \delta T}{\partial x_i} + \delta u_i\frac{\partial T^{(0)}}{\partial x_i}\right) + n^{(0)}\left(T^{(0)}\frac{\partial \delta u_i}{\partial x_i} + \delta T\frac{\partial u_i^{(0)}}{\partial x_i}\right) = 0 \tag{4.28}$$

where

$$\delta\rho = m\delta n \tag{4.29}$$

and

$$\delta P = n^{(0)}\delta T + T^{(0)}\delta n \tag{4.30}$$

are the contributions to the mass density and to the pressure from the second approximation, respectively. The steady field equations (4.26), (4.27), and (4.28) that follow from the second approximation are linear in δn, δT, and $\delta\mathbf{u}$ and can be easily solved provided that one knows the solution from the first approximation. Thus, the solution of a steady gas flow problem by Hilbert's method requires first the solution of Euler's equations. The effects of viscosity and thermal conductivity can then be taken into account by solving the above linear steady field equations. Therefore, the effect of boundary layer or shock layer in the flow should be considered in the second approximation.

5. HIGHER APPROXIMATIONS (THE BURNETT EQUATIONS, ETC.)

Considering the next approximation [Eq. (2.15)] for steady flow, one has

$$v_i\frac{\partial}{\partial x_i}\left(f_L\phi^{(1)}\right) = -n^{(0)}f_L\left[I\phi^{(2)} + J_s(\phi^{(1)},\phi^{(1)})\right] \tag{5.1}$$

Substitution from Eqs. (3.4) and (4.6) for f_L and $\phi^{(1)}$ into Eq. (5.1) after cumbersome manipulations yields

$$
\begin{aligned}
&-n^{(0)}I\phi^{(2)}\\
&\quad= v_k\Bigg[\frac{1}{n^{(0)}T^{(0)^2}}\left(\frac{2\mathscr{v}_i}{T^{(0)}}-\frac{\partial\mathscr{v}_i}{\partial T^{(0)}}\right)\frac{\partial T^{(0)}}{\partial x_i}\,\frac{\partial T^{(0)}}{\partial x_k}+\frac{1}{n^{(0)^2}T^{(0)^2}}\mathscr{v}_i\frac{\partial T^{(0)}}{\partial x_i}\,\frac{\partial n^{(0)}}{\partial x_k}\\
&\qquad+\frac{1}{n^{(0)^2}T^{(0)^2}}\tau_{ij}D_{ij}\frac{\partial n^{(0)}}{\partial x_k}-\frac{1}{n^{(0)}T^{(0)^2}}\mathscr{v}_i\frac{\partial^2 T^{(0)}}{\partial x_i\partial x_k}-\frac{1}{n^{(0)}T^{(0)}}\tau_{ij}\frac{\partial D_{ij}}{\partial x_k}\\
&\qquad-\frac{1}{n^{(0)}T^{(0)}}\left(\frac{\partial\tau_{ij}}{\partial T^{(0)}}-\frac{\tau_{ij}}{T^{(0)}}\right)D_{ij}\frac{\partial T^{(0)}}{\partial x_k}-\frac{1}{n^{(0)^2}}\frac{\partial n^{(0)}}{\partial x_k}\delta n\\
&\qquad+\left(\frac{5}{2}\,\frac{1}{T^{(0)^2}}\,\frac{\partial T^{(0)}}{\partial x_k}+\frac{1}{T^{(0)^2}}\,\frac{\partial\epsilon}{\partial x_k}-\frac{2\epsilon}{T^{(0)^3}}\,\frac{\partial T^{(0)}}{\partial x_k}\right)\delta T\\
&\qquad+\frac{m}{T^{(0)}}\,\frac{\partial u_i^{(0)}}{\partial x_k}\delta u_i+\frac{1}{n^{(0)}}\,\frac{\partial(\delta n)}{\partial x_k}+\frac{1}{T^{(0)^2}}\left(\epsilon-\frac{5}{2}T^{(0)}\right)\frac{\partial(\delta T)}{\partial x_k}\\
&\qquad+\frac{m}{T^{(0)}}\left(v_i-u_i^{(0)}\right)\frac{\partial(\delta u_i)}{\partial x_k}\Bigg]\\
&\quad+\left(\frac{1}{n^{(0)}T^{(0)^2}}\mathscr{v}_i\frac{\partial T^{(0)}}{\partial x_i}+\frac{1}{n^{(0)}T^{(0)}}\tau_{ij}D_{ij}-\psi^{(1)}\right)\\
&\quad\times\Bigg[\frac{1}{T^{(0)^2}}\left(\epsilon-\frac{5}{2}T^{(0)}\right)\left(v_l-u_l^{(0)}\right)\frac{\partial T^{(0)}}{x_l}\\
&\qquad+\frac{m}{T^{(0)}}\left[\left(v_s-u_s^{(0)}\right)\left(v_r-u_r^{(0)}\right)-\frac{1}{3}\delta_{sr}(\mathbf{v}-\mathbf{u})^2\right]\frac{\partial u_s^{(0)}}{\partial x_r}\Bigg]\\
&\quad+n^{(0)}\Bigg[\frac{1}{n^{(0)^2}T^{(0)^4}}J_s\left(\mathscr{v}_i,\mathscr{v}_j\right)\frac{\partial T^{(0)}}{\partial x_i}\,\frac{\partial T^{(0)}}{\partial x_j}\\
&\qquad+\frac{1}{n^{(0)^2}T^{(0)^3}}J_s\left(\mathscr{v}_i,\tau_{kl}\right)\frac{\partial T^{(0)}}{\partial x_i}D_{kl}-\frac{1}{n^{(0)}T^{(0)^2}}J_s\left(\mathscr{v}_i,\psi^{(1)}\right)\\
&\qquad+\frac{1}{n^{(0)^2}T^{(0)^3}}J_s\left(\tau_{ij},\mathscr{v}_l\right)D_{ij}\frac{\partial T^{(0)}}{\partial x_l}+\frac{1}{n^{(0)^2}T^{(0)^2}}J_s\left(\tau_{ij},\tau_{kl}\right)D_{ij}D_{kl}\\
&\qquad-\frac{1}{n^{(0)}T^{(0)}}J_s\left(\tau_{ij},\psi^{(1)}\right)D_{ij}-\frac{1}{n^{(0)}T^{(0)^2}}J_s\left(\psi^{(1)},\mathscr{v}_j\right)\frac{\partial T^{(0)}}{\partial x_j}\\
&\qquad-\frac{1}{n^{(0)}T^{(0)}}J_s\left(\psi^{(1)},\tau_{kl}\right)D_{kl}+J_s\left(\psi^{(1)},\psi^{(1)}\right)\Bigg]
\end{aligned}
\tag{5.2}
$$

The solution of the linear integral equation (5.2) for $\phi^{(2)}$ will consist of terms up to second derivatives in $\mathbf{u}^{(0)}$ and $T^{(0)}$, up to first derivatives in $n^{(0)}$, $\delta\mathbf{u}$, δn, and δT, combinations of them, plus a linear form $\psi^{(2)}$ of the summational invariants

$$\psi^{(2)} = C_1' + C_\epsilon' \epsilon + C_i'\left(v_i - u_i^{(0)}\right) \tag{5.3}$$

where C_1', C_ϵ', and C_i' depend only on the coordinates x, y, and z and are related to the hydrodynamic variables $\delta n'$, $\delta\mathbf{u}'$, and $\delta T'$ through relations (2.7), (2.8), and (2.9), which reduce to

$$\delta n' = \int f_L \phi^{(2)}\, d^3v \tag{5.4}$$

$$n^{(0)}\delta u_i' = \int \left(v_i - u_i^{(0)}\right) f_L \phi^{(2)}\, d^3v - \delta n \delta u_i \tag{5.5}$$

$$\tfrac{3}{2}\left(n^{(0)}\delta T' + T^{(0)}\delta n'\right) = \int \epsilon f_L \phi^{(2)}\, d^3v - \tfrac{3}{2}\delta n \delta T - \tfrac{1}{2}\rho^{(0)}\delta u_i \delta u_i \tag{5.6}$$

in this approximation. The solvability conditions for steady flow

$$\int \psi_\alpha v_i \frac{\partial}{\partial x_i} f_L \phi^{(2)}\, d^3v = 0 \tag{5.7}$$

together with Eqs. (5.4), (5.5), and (5.6), will then yield the steady hydrodynamic equations (the Burnett equations) in this approximation.

Higher approximations can be carried out similarly.

6. SUMMARY AND CONCLUSIONS

The Hilbert expansion to the Boltzmann equation is carried out for steady flow. As was shown by Hilbert, the first term in the series for the distribution function is a local Maxwellian; thus, the derived hydrodynamic equations in the first approximation are Euler's equations. The second approximation for the distribution function in the Hilbert series is given by Eq. (4.6), and the steady field equations that follow from the second approximation are derived. The coefficients of thermal conductivity and viscosity of Hilbert that appear in the latter equations are explicitly given by Eqs. (4.25) and (4.22), respectively, and are seen to be precisely the same as those obtained by Chapman and Enskog. The procedure to obtain higher approximations is also summarized.

It is clearly seen that the microscopic solution leads to the macroscopic hydrodynamic equations of the required order (Euler, Navier–Stokes, Burnett, and etc.) with appropriate boundary conditions. The initial value problem has been excluded from the discussion by restricting the expansion to steady flow.

The Hilbert solution of a steady gas flow problem can thus be obtained by first solving the steady Euler equations of ideal gas flow and then, the linear steady field equations derived in Section 4 with appropriate boundary conditions. The effects of viscosity and thermal conductivity in steady gas flow should be taken into account in the solution of the steady field equations of the second approximation.

After the paper had been sent for publication, the author learned that Dr. R. G. Muncaster(13) also treated the Hilbert expansion in detail.

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APPENDIX: TABLE OF INTEGRALS

$$\int f_L \psi^{(1)}\, d^3v = n^{(0)} C_1 + \tfrac{3}{2} n^{(0)} T^{(0)} C_\epsilon \qquad (\mathrm{A1})$$

$$\int (v_i - u_i^{(0)}) f_L \psi^{(1)}\, d^3v = n^{(0)} \frac{T^{(0)}}{m} C_i \qquad (\mathrm{A2})$$

$$\int (v_i - u_i^{(0)})(v_j - u_j^{(0)}) f_L \psi^{(1)}\, d^3v = \left(n^{(0)} \frac{T^{(0)}}{m} C_1 + \frac{5}{2} n^{(0)} \frac{T^{(0)^2}}{m} C_\epsilon \right) \delta_{ij} \qquad (\mathrm{A3})$$

$$\int (v_i - u_i^{(0)})(v_j - u_j^{(0)}) f_L \gamma_k\, d^3v = 0 \qquad (\mathrm{A4})$$

$$\int (v_i - u_i^{(0)})(v_j - u_j^{(0)}) f_L \tau_{kl}\, d^3v = \eta \frac{n^{(0)} T^{(0)}}{m} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3} \delta_{ij}\delta_{kl} \right) (\mathrm{A5})^4$$

$$\int \epsilon f_L \psi^{(1)}\, d^3v = \frac{3}{2} n^{(0)} T^{(0)} C_1 + \frac{15}{4} n^{(0)} T^{(0)^2} C_\epsilon \qquad (\mathrm{A6})$$

$$\int \epsilon (v_i - u_i^{(0)}) f_L \psi^{(1)}\, d^3v = \frac{5}{2} n^{(0)} \frac{T^{(0)^2}}{m} C_i \qquad (\mathrm{A7})$$

$$\int \epsilon (v_i - u_i^{(0)}) f_L \gamma_j\, d^3v = \lambda n^{(0)} T^{(0)^2} \delta_{ij} \qquad (\mathrm{A8})^5$$

$$\int \epsilon (v_i - u_i^{(0)}) f_L \tau_{kl}\, d^3v = 0 \qquad (\mathrm{A9})$$

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[4] η is the dynamic viscosity.

[5] λ is the thermal conductivity.

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